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THE DYNAMICS OF A THIN SLUG OF REACTING IMPURITY IN FLOW IN A POROUS MEDIUM†

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An attempt is made to establish the relation between the initial concentration distribution of an impurity in a slug and the asymptotic behaviour of this distribution under the assumption that the impurity is transported by uniform seepage (rectilinear-parallel or plane-parallel) in a homogeneous porous medium where sorption and chemical reaction occur. The carrying liquid is assumed to be homogeneous and incompressible and the effect of the impurity on the flow is ignored (a passive impurity). Problems with localized initial conditions, their qualitative properties and self-similar asymptotic forms are studied. The principal step in the evolution of a thin slug of a reacting impurity turns out to be quite simple: a concentration distribution is close in form to the corresponding distribution in the problem without reaction but with a variable amplitude; the evolution dynamics of the amplitude and the pulse width are rather well predicted by the method of integral relations. The possibility of using the solutions obtained and of the approaches developed to analyse the effect of the absorption of the active impurity during the reaction on the effectiveness of increasing the yield of petroleum by the injection of thin slugs of an active impurity is discussed.

The problem of a thin slug, that is, the propagation and evolution of a batch of impurity, initially localized in a narrow domain, has at least two important applications: ecological (the spreading of local contamination of ground waters) and petroleum production (the use of thin slugs of reagents to improve the yield of petroleum) [1–4]. The available theoretical results relate to the purely diffusive evolution of a slug [3, 5] and to the effect of the sorption irreversibility [6–9].

In this paper, which is a direct development of these investigations, particular attention is paid to the effect of a reaction which leads to the absorption of the impurity on the evolution of the slug.

1. FORMULATION OF THE PROBLEM

Consider the transport of a neutral impurity by the flow of an incompressible fluid in a porous medium when there is adsorption, which is subsequently assumed to be linear and at equilibrium, and an nth order chemical reaction. Rectilinear-parallel and plane-radial flows are investigated (cases 1 and 2, respectively). Here, the concentration distribution of the impurity is governed by the diffusion equation with convection and reaction, which, for the two cases mentioned above, has the form

$$\frac{\partial(m+\Gamma)c}{\partial t}+U\frac{\partial c}{\partial x}=D\frac{\partial^2 c}{\partial x^2}-\alpha c^n \text{ (case 1)} \tag{1.1}$$

$$m\frac{\partial(m+\Gamma)c}{\partial t}+\frac{Q}{2\pi r}\frac{\partial c}{\partial r}=D\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial c}{\partial r}-\alpha c^n \text{ (case 2)} \tag{1.2}$$

Here, m is the porosity of the medium, D is the diffusion coefficient for the impurity, U is the seepage rate, Q is the seepage rate per unit thickness of the stratum, α is the reaction rate constant, n is the order of the reaction and Γ is Henry's constant for adsorption. All these quantities are assumed to be positive constants.

In this case, the amount of the impurity that is sorbed per unit volume is

$$a=\Gamma c \tag{1.3}$$

We consider the initial-value problem with initial distribution of the form

$$c(x,0)=C_0(x)\geq 0,\ |x|\leq\varepsilon,\ \ c(x,0)=0,\ |x|>\varepsilon \text{ (case 1)} \tag{1.4}$$

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$$c(r,0)=C_0(r)>0,\ \ 0\le r\le\varepsilon,\ \ c(r,0)=0,\ \ r>\varepsilon \text{ (case 2)} \tag{1.5}$$

where ε is the initial size of the slug, which is assumed to be small in a certain sense (see below).

The boundary conditions are that a solution is sought which decays at infinity and (in the radial case) there is no inflow of the impurity from a source at the origin of the system of coordinates

$$r=0:\ -2\pi Dr\frac{\partial c}{\partial r}+Qc=0 \tag{1.6}$$

It is well known that, by changing to a moving system of coordinates, the problem for rectilinear-parallel flow with a linear sorption isotherm can be reduced to a diffusion equation with reaction, but without convection, which formally corresponds to $m = 1$, $U = 0$ and $\Gamma = 0$.

The simplest prototype of the formulation of the problem is obtained if one considers the usual diffusion equation with initial data of the form (1.4). We then have the expression

$$c(x,t)\approx\frac{M_0}{\sqrt{\pi Dt}}\exp\left(-\frac{x^2}{4Dt}\right),\ \ t>>\frac{\varepsilon^2}{4D};\ \ M_0=\int_{-\varepsilon}^{\varepsilon}C_0(x)dx \tag{1.7}$$

for the asymptotic form of the concentration distribution in a pulse in terms of a single integral characteristic of the initial distribution, that is, of the total initial amount of the impurity M_0.

The aim of this paper is to seek an analogue of relation (1.7) for the non-linear problem. It is well known that the result cannot be so simple and general and cannot be expressed in terms of just a single initial amount because the amount of impurity is not conserved when there is a chemical reaction and its rate of decrease depends on the form of the initial distribution.

2. THE SELF-SIMILAR SOLUTION. LINEAR FLOW

We will consider the diffusion equation with a reaction

$$\frac{\partial c}{\partial t}=D\frac{\partial^2 c}{\partial x^2}-\alpha c^n\ \ \alpha>0 \tag{2.1}$$

and we shall analyse the evolution of an initial perturbation

$$c(x,0)=F(x)>0,\ \ |x|\le\varepsilon;\ \ c(x,0)=0,\ \ |x|>\varepsilon \tag{2.2}$$

which is localized in a small domain $|x|\leqslant\varepsilon$.

It is natural to expect that it will asymptotically generate a self-similar asymptotic form after long periods of time.

In the unbounded domain, $\infty<x<\infty$, Eq. (2.1) admits of self-similar solutions of the form

$$c=Bt^{-\gamma}f(\xi),\ \ \xi=\frac{x}{2\sqrt{Dt}},\ \ \gamma=\frac{1}{n-1} \tag{2.3}$$

We also choose $B^{n-1}\alpha = 1$. Then, the required solution must satisfy the boundary-value problem

$$f''+2\xi f'-4f^n+4\gamma f=0,\ \ 0\le\xi<\infty;\ \ f'(0)=0,\ \ f(\infty)=0 \tag{2.4}$$

It is clear that, in the case of the chosen (positive) initial data, the solution must be non-negative and clearly cannot have a maximum exceeding the critical value $f^*(n) = (n-1)^\gamma$. Actually, at the point of the maximum $f' = 0, f'' < 0$ and, according to (2.4), $f < f^*$. Hence, the required solutions can begin at points of the interval $x = 0, 0 \leqslant f(0) \leqslant f^*(n)$ with $f'(0) = 0$ and are easily calculated numerically as solutions of the corresponding Cauchy problem. Typical results for $n = 1.5$, 2 and 3 are shown in Fig. 1 for various values of the amplitude parameter $A = f(0)$.

These results deserve some specific commentary. First, solutions of the required form, that is, positive solutions, which have a maximum at $x = 0$ and tend to zero at infinity exist for a certain range of values of the amplitude parameter A.

Moreover, the existence of a self-similar solution, generally speaking, does not mean that it will express the required asymptotic form. On the contrary, when $n \geqslant 3$, the opposite is directly proved: when $n \geqslant 3$, the required asymptotic forms are clearly not expressed by the self-similar solution of (2.3) and (2.4).

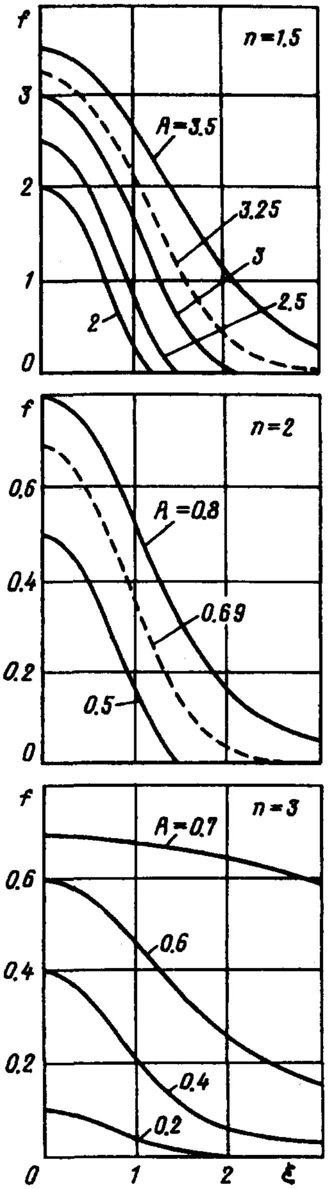

Fig. 1.

This manifests itself in a particularly distinct manner when $n = 3$. Actually, the total amount of the substance is expressed by the integral

$$M(t) = \int_{-\infty}^{\infty} c(x,t)dx$$

and, according to the physical meaning of the problem, it must decrease (which is easily shown by integrating Eq. (2.1) over the whole of the x axis).

In the case of a self-similar solution of the form (2.3), we have

$$M(t) = Bt^{-\gamma}\int_{-\infty}^{\infty} f\left(\frac{x}{2\sqrt{Dt}}\right)dx = 2B\sqrt{Dt}^{-\gamma+\frac{1}{2}}\int_{-\infty}^{\infty} f(\xi)\,d\xi$$

and this expression does not decrease if $-\gamma + 1/2 \geqslant 0$ or $n \geqslant 3$. In particular, when $n = 3$, "the supply of the substance", which corresponds to the self-similar solution, must remain constant. (The discussion relating to the analysis of the total supply is due to G. I. Barenblatt.)

This situation, that appears to be paradoxical at a first glance, is resolved by a more profound analysis of the self-similar solutions of problem (2.4). As previously, we shall assume that $n > 1$. Then, the asymptotic form of the solutions which decay at infinity is close to the solutions of the linear equation

$$\varphi'' + 2\xi\varphi' + 4\gamma\varphi = 0 \tag{2.5}$$

The two linearly independent solutions of Eq. (2.5) behave as

$$\varphi_1 = \exp(-\xi^2),\ \varphi_2 = \xi^{-2\gamma} \tag{2.6}$$

When $n > 3$, an infinite supply of the substance corresponds to all solutions of problem (2.4), which contain a slowly decaying contribution φ_2 in the asymptotic form.

These solutions can only serve as the asymptotic forms of solutions of problems with an initial infinite supply of the impurity and non-localized initial data. Solutions of problem (2.4), which decay exponentially at infinity, change sign and cannot serve as the asymptotic form of the solutions of problems with positive initial data.

When $1 < n < 3$, both solutions (2.6) of the linearized equation (2.5) are integrable at infinity and there is no paradox associated with an infinite supply. Nevertheless, also in this case, solutions with exponential asymptotic forms at infinity cannot serve as a self-similar asymptotic form of problems with localized (finite) initial data. This intuitive discussion is supported by rigorous "a priori" estimates (see Section 3).

Hence, only those positive solutions of (2.4) can serve as possible self-similar asymptotic forms that decay more rapidly than any power of ξ as $\xi \to \infty$.

This condition enables one to isolate from the single-parameter family of solutions of problem (2.4) the single solution that can serve as a "candidate" for the role of self-similar asymptotic form which is, in fact, the positive solution which decays as $\xi \to \infty$ more rapidly than $\xi^{-2/(n-1)}$.

In fact, the question refers to the determination of the minimum value of the amplitude parameter $A = f(0)$ for which the solution still remains positive when $0 \leqslant \xi < \infty$. This value is easily found numerically. It depends on n and is a unique eigenvalue of the problem. The corresponding solutions are represented by the dashed line in Fig. 1. The arguments presented above enable one to hope that the self-similar solution that has been chosen in this manner is the asymptotic form of the solution in the case of a localized initial distribution. This supposition is well supported by the results of a numerical experiment.

Comparison of the normalized results of a calculation of the concentrations $c(x, t)/c(0, t)$ as a function of the self-similar coordinate $\xi = x/(2\sqrt{(Dt)})$ and the normalized self-similar solution $f_n(\xi)/A(n)$ shows that these solutions are the same to within a few percent over the whole range of ξ values. (The numerical solution was obtained for the initial concentration profile n the form of a narrow rectangle of unit height.) Hence, when $n < 3$, the asymptotic behaviour of the concentration distribution is characterized by the relation

$$c(x,t) \sim Bt^{-\gamma} f_n\left(\frac{x}{2\sqrt{Dt}}\right),\ t \gg \frac{l^2}{4D}$$

where B is the functional of the initial distribution $C_0(x)$.

In the case of standard initial distributions, the form of this functional can be established from dimensional considerations. In fact, say, for a rectangular initial distribution: $C_0(x) = C_0$, $|x| < l$, $C_0(x) = 0$, $|x| > l$, the coefficient B can be a function of C_0, l, D and α.

Hence

$$B = \alpha^{-\gamma}\Phi(Z),\ Z = C_0\left(l^2\alpha / D\right)^{\gamma}$$

It follows that initial distributions with the same initial value of the moment $M^* = C_0 l^{2\gamma}$ lead to the same asymptotic profiles. This conclusion is confirmed by data from a numerical experiment for certain values of the reaction rate α, initial concentration C_0 and initial pulse width l but the same value of the dimensionless parameter $Z = C_0 l^2\alpha/D$. The results are practically identical and are in good agreement with the self-similar distribution.

We emphasize that, as would be expected, the narrower the initial pulse, the greater the initial supply of the impurity which is necessary to obtain one and the same asymptotic behaviour and, as $l \to 0$, this supply tends to infinity.

Actually, this means that, in the case of narrow pulses, the greater part of the supply is consumed in the reaction at the initial stage of the evolution.

Hence, in view of the fact that there is no invariant of the motion in the case of this problem, the self-similar asymptotic form retains a certain non-trivial "memory" about the form, amplitude and width of the initial pulse, a situation which is typical for the self-similarity of the second kind [10].

3. "A PRIORI" ESTIMATES

We will now consider the bounded solution $c(x, t)$ of problem (2.1) and (2.2). Suppose $x^*(t)$ is the position of the global maximum of the solution at the instant of time t

$$c(x, t) \leq c(x^*(t), t) \equiv c_*(t) \tag{3.1}$$

It is obvious that

$$0 \leq c_*; \ \frac{dc}{dx}\Big|_{x=x^*} = 0, \ \frac{d^2c}{dx^2}\Big|_{x=x^*} \leq 0 \tag{3.2}$$

Hence

$$\begin{gathered} \frac{dc_*(t)}{dt} = D\frac{d^2c}{dx^2}\Big|_{x=x^*} - \alpha c^n \leq -\alpha c_*^n \\ c_*(t) \leq \left[c_*^{1-n}(0) + (n-1)\alpha t\right]^{-\gamma} = \hat{c}_*(t) \end{gathered} \tag{3.3}$$

The first inequality of (3.3) expresses the obvious fact that a decrease in the maximum concentration occurs not only due to reaction at the given point but also due to the fact that the substance flows away from the neighbourhood of the maximum into a region with a lower concentration. It is noteworthy that, when $n > 1$, the asymptotic behaviour of $\hat{c}_*(t)$ is independent of $c_*(0)$ and is given by the limit dependence

$$c^0(t) = [(n-1)\alpha t]^{-\gamma} \tag{3.4}$$

The upper estimate $\hat{c}_*(t)$ is found to be below the limit estimate.

It is easily proved [11, 12] that the solution depends monotonically on the reaction rate and the initial data and is majorized by the solution of the problem without reaction. In particular, it follows that the solution of a problem with finite initial data decays more rapidly at infinity than any power of a coordinate, which is important when choosing the self-similar solution.

We consider a linear equation with a variable reaction coefficient

$$\frac{\partial \tilde{c}}{\partial t} = D\frac{\partial^2 \tilde{c}}{\partial x^2} - \alpha c_m^{n-1}(t)\tilde{c}, \ c_m(t) = \max_x c_0(x,t) \tag{3.5}$$

Here $c_0(x, t)$ is the solution of the diffusion equation without reaction with the same initial conditions. Equation (3.5) has the solution

$$\tilde{c}(x,t) = R(t)c_0(x,t), \ R(t) = \exp\left(-\alpha \int_0^t c_m^{n-1}(t')dt'\right) \tag{3.6}$$

that gives a lower estimate in the case of the required solution of the diffusion problem without reaction for the same initial data.

Hence, the solution of the non-linear problem turns out to be bounded on two sides by the two solutions of the linear problems

$$c_0(x, t) \geq c(x, t) \geq R(t)c_0(x, t) \tag{3.7}$$

It is essential that in the case of finite initial data

$$c_m(t) \sim M_0(\pi D t)^{-1/2}; \text{ when } t \to \infty \tag{3.8}$$

Hence, when $n > 3$, the factor $R(t)$ tends to a finite limit as $t \to \infty$. In particular, when $n > 3$, the total supply of the impurity in the slug remains bounded as $t \to \infty$

$$M_0(t) \geq M(t) = \int_{-\infty}^{\infty} c(x,t)dx \geq R(\infty)M_0(t) \tag{3.9}$$

It can be postulated that the solution of the non-linear problem tends asymptotically to the solution of the linear problem with a smaller total supply. This assumption is reinforced by the results of a numerical experiment and the approximate solution of the problem by the method of integral relations (see below).

All the preceding discussions can be repeated with minimum modifications for radially symmetric flow with convection. The only important difference is the fact that, in the asymptotic form (3.7), the power exponent is replaced by −1. The value $n = 2$, therefore, turns out to be critical. When $n > 2$, the supply of impurity decreases and tends to a finite limit, and it is therefore natural to suppose that the asymptotic form of the solution will be the same as the asymptotic form of the solution of the linear problem. It once again turns out that this conclusion is in good agreement with numerical experiment.

Remark. Some results in Sections 2 and 3 are included among the more general results in [11, 12].

4. RECTILINEAR-PARALLEL FLOW. THE METHOD OF INTEGRAL RELATIONS

It is clear from the analysis of the results of the numerical experiment and the asymptotic investigation that the solution being investigated is close to a standard Gaussian curve of variable amplitude and width. It therefore makes sense to seek an approximate solution of problem (1.1)–(1.4) in the form

$$c(x, t) = B(t)\exp(-x^2/l^2) \tag{4.1}$$

where $B(t)$ and $l(t)$ are unknown functions of time. We will use the method of integral relations (see [1, 2], for example). On multiplying both sides of Eq. (2.1) by c_k, integrating from $-\infty$ to ∞ and taking account of the condition at infinity, we obtain the integral relations

$$\begin{gathered}\int_{-\infty}^{\infty} c^k \frac{dc}{dt}dx = \frac{1}{k+1}\frac{d}{dt}\int_{-\infty}^{\infty} c^{k+1}dx = \\ = \int_{-\infty}^{\infty} c^k \frac{d^2c}{dx^2}dx - \alpha\int_{-\infty}^{\infty} c^{k+n}dx = -k\int_{-\infty}^{\infty} c^{k-1}\left(\frac{dc}{dx}\right)^2 dx - \alpha\int_{-\infty}^{\infty} c^{k+n}dx\end{gathered} \tag{4.2}$$

for finding the parameters B and l. It is essential that when choosing the approximation in the form of (4.1), the integrals (4.2) are expressed in an explicit form in terms of B and l. Using the first two integral relations ($k = 0, 1$) and putting $Bl = X$, we have

$$\frac{dX}{dt} = -\frac{\alpha}{\sqrt{n}}XB^{n-1}, \quad \frac{dB}{dt} = -\frac{2B^3}{X^2} + \alpha B^n\left(\frac{1}{\sqrt{n}} - \frac{2\sqrt{2}}{\sqrt{n+1}}\right) \tag{4.3}$$

Eliminating t from system (4.3), we obtain the equation for the trajectories in the phase plane of the variables B and X

$$\frac{dB}{dX} = \frac{2\sqrt{n}B^{4-n}}{X^3} - \frac{B}{X}\left(1 - 2\sqrt{\frac{2n}{n+1}}\right) \tag{4.4}$$

The solutions of interest have an amplitude that decreases with time. In the case of a small B and finite X, the behaviour of the solutions of Eq. (4.4) is different when $n < 3$ and $n > 3$. If $n < 3$, the second term on the right-hand side is the leading term and we, therefore have $X \to 0$ as $B \to 0$.

Thus, in the case of asymptotically decaying solutions, both the parameters, B and X, tend to zero and the leading term on the right-hand side of (4.4) turns out to be the first term. Hence

$$\frac{dB}{dX} \sim 2\sqrt{n}\frac{B^{4-n}}{X^3}, \quad X \sim [\sqrt{n}(n-3)]^{1/2}B^{(3-n)/2} \tag{4.5}$$

The first equation of (4.3) then gives

$$t \sim B^{1-n}, \quad l = X/B \sim B^{(1-n)/2} \sim t^{1/2} \tag{4.6}$$

which agrees with the behaviour expected from the analysis of the self-similar solution.

When $n > 3$, the trajectories tend to finite points on the X axis as $B \to 0$. In this case, it follows from (4.3) that $B \sim t^{-1/2}$, $l \sim t^{1/2}$. This corresponds to the asymptotic form of the solutions of the linear problem. In the boundary case $n = 3$, Eq. (4.5) becomes linear and is easily integrated

$$\ln B = \text{const} - \frac{\sqrt{3}}{\alpha}\frac{1}{X^2} + (\sqrt{6}-1)\ln X \tag{4.7}$$

After this, the problem is completely solved in quadratures. On substituting expression (4.7) into the first equation of (4.3) and integrating, we have

$$B = \text{const}\, X^{(\sqrt{6}-1)} \exp\left(-\frac{\sqrt{3}}{\alpha X^2}\right)$$
$$t = \frac{\sqrt{3}}{\alpha(\text{const})^2}\int_X^{X_0} X^{1-2\sqrt{6}} \exp\left(\frac{2\sqrt{3}}{\alpha X^2}\right) dX \tag{4.8}$$

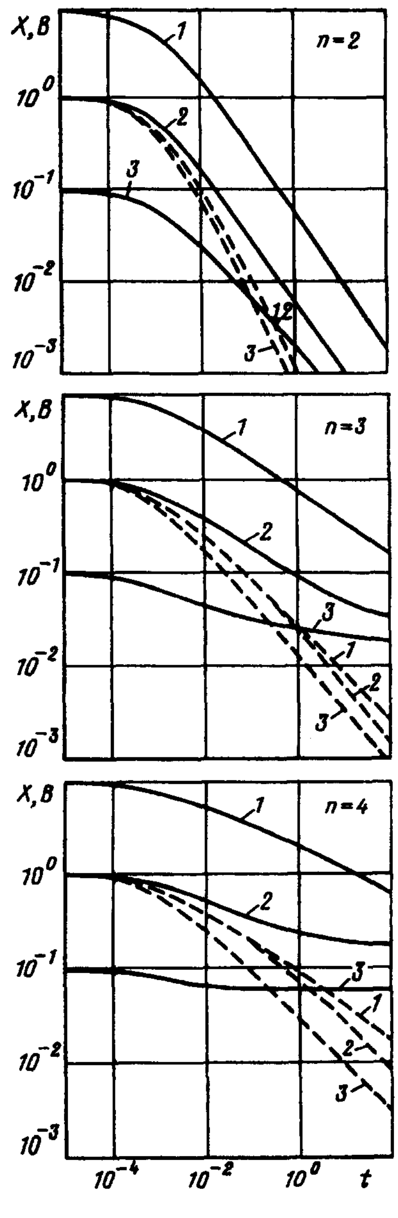

Fig. 2.

The latter expression tends extremely rapidly to infinity, this signifies a very slow decay of the parameter X with time, which is proportional to the total supply of the substance in the slug, and in this case

$$X \sim [2\sqrt{3}/(\alpha \ln t)]^{1/2}$$

On the other hand, by comparing expressions (4.8), we find that $B\sqrt{t} \sim X$. It follows that the characteristic width of the slug $l = X/B$ behaves asymptotically in accordance with the usual square root law ($l \sim \sqrt{t}$) and the amplitude decays somewhat more rapidly according to the square root law

$$B \sim X/\sqrt{t} \approx [2\sqrt{3}/(\alpha t \ln t)]^{1/2}$$

Figure 2 illustrates the behaviour of the specific parameters of the slug: the total supply $X = Bl$ (the dashed curves) and the amplitude B (the solid curves) in the subcritical case ($n = 2$), the supercritical case ($n = 4$) and the boundary case ($n = 3$) when $\alpha = 10^3$, $X(0) = 1$ and for various initial values of the amplitude B_0 equal to 10 (curve 1), 1 (curve 2) and 10^{-1} (curve 3). The approximate solution $X(t)$, obtained by the method of integral relations (MIR) for $n = 3$ (the solid curve), is compared with the solution found from the numerical experiment (the dashed curve). Since the initial distribution (a rectangle) in the numerical calculation was extremely far from that postulated in the MIR, the initial values of B and l in the comparison were taken to be such that the same values of the first and second moments were obtained as in the initial distribution. MIR provides a reasonable approximation to the solution over the whole range of observations. It even reproduces the rather unusual dynamics of the total supply in the boundary case when $n = 3$.

The distributions obtained numerically, predicted by MIR and those corresponding to the self-similar solution, were also compared. They are found to be extremely close to one another, that justifies the choice of the form of the solution in the MIR.

This all favours the use of MIR for practical estimates.

5. RADIAL FLOW

The linear case. In the linear case ($n = 1$), on making the substitution $c = \hat{c}(r, t)e^{-\alpha t}$, the transport equation is reduced to the diffusion equation without absorption. The solution of this equation, corresponding to initial data that are localized close to the point $r = 0$, and to zero boundary conditions when $r = 0$ and $r = \infty$, has the form

$$\hat{c} = \frac{M}{t} f(\eta), \quad \eta = \frac{r}{2\sqrt{Dt}}$$

$$f'' + \left(2\eta + (1 - q)\frac{1}{\eta}\right) f' + 4f = 0; \quad q = \frac{Q}{2\pi D}$$

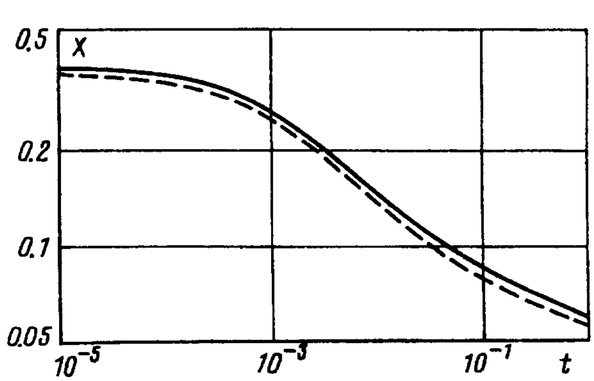

Fig. 3.

The solution of the equation for the function $f(\eta)$ which decays more rapidly than η^{-2} as $n \to \infty$ (which is necessary for the convergence of the integral of $\eta f(\eta)$, that is proportional to the total supply of the impurity in the slug) has the form [13]

$$f = C_1\eta^q \exp(-\eta^2)d\eta, \quad C_1 = \frac{1}{\pi}\left[\Gamma\left(\frac{1+q}{2}\right)\right]^{-1}$$

The constant C_1 is found from the normalization condition

$$2\pi\int_0^\infty \eta f(\eta)d\eta = 1$$

This solution illustrates the important properties of the evolution of a slug close to a source ("a pore") when there is convection. Close to the source ($r = 0$), the concentration variation is given by a power law with an exponent which depends on the Péclet number of the problem (the dimensionless parameter q), that specifies the ratio of the convective and diffusive fluxes.

The concentration vanishes at the source itself, and the slug is an annular domain which is narrower, the greater the Péclet number q. All of these properties are also conserved in the non-linear case, which is considered below.

The non-linear case—the self-similar solution. In the non-linear case formulated in Section 1, for radial flow the problem has a self-similar solution of the form

$$C = Bt^{-\gamma}f(\eta), \quad \gamma = \frac{1}{n-1}, \quad \eta = \frac{r}{2\sqrt{Dt}} \tag{5.1}$$

where f is the solution of the boundary-value problem

$$f'' + \left[2\eta + (4-q)\frac{1}{\eta}\right]f' + 4\gamma f - 4f^n = 0, \quad f(0) = f(\infty) = 0 \tag{5.2}$$

In accordance with its physical meaning, the required solution must be positive and therefore has a positive maximum. It is well known that there is no such solution when $n < 1$. When $n > 1$, as the direct numerical solution of problem (5.2) shows, the self-similar solution has the required form in a certain range of the parameter values

$$A = \lim_{\eta\to 0}(\eta^q f(\eta))$$

The corresponding "total supply" of the impurity

$$M = \int_0^\infty 2\pi rc(r, t)dr = \frac{8\pi DA}{t^{\gamma-1}}\int_0^\infty \eta f(\eta)d\eta \tag{5.3}$$

does not decay when $n \geqslant 2$. The self-similar solution therefore cannot be the asymptotic form of the solution with localized initial data when $n \geqslant 2$.

On the whole, the situation is entirely equivalent to the case of a rectilinear-parallel flow with the sole difference that the range of values of the exponent n in which the asymptotic form can be self-similar is narrowed down to the range $1 < n < 2$. In this range, the unique solution is singled out by the fact that it decays like $\exp(-\eta^2)$ as $\eta \to \infty$ since the "total supply" is only bounded for such a solution. A comparison of the self-similar solutions with the asymptotic form of the numerical solution confirms that the self-similar solution actually describes the asymptotic form of the evolution of the concentration in the slug in this range of values of the parameter n.

Again, it is easy to prove the bilateral estimate

$$\tilde{c}(r, t) \leqslant c(r, t) \leqslant c_0(r, t)$$

where $c_0(r, t)$ is the solution of the corresponding linear problem without absorption ($\alpha \equiv 0$) and $\tilde{c}$ is the solution of the linearized problem

$$\frac{\partial\tilde{c}}{\partial t} + \frac{Q}{2\pi r}\frac{\partial\tilde{c}}{\partial r} = D\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\tilde{c}}{\partial r}\right) - \alpha c_m^{n-1}(t)\tilde{c}, \quad c_m(t) = \max_r c_0(r, t)$$

$$\tilde{c}(0,\ t)=\tilde{c}(\infty,\ t)=0;\ \ \tilde{c}(r,\ 0)=\varphi_0(r),\ \ r\leqslant\varepsilon;\ \ \tilde{c}(r,\ 0)=0,\ \ r>\varepsilon$$

$$\tilde{c}(r,\ t)=c_0(r,\ t)R(t)$$

When $n > 2$, the factor $R(t)$, which is determined by relation (3.6), tends to a finite limit as $t \to \infty$ such that the total supply of the impurity tends to a finite limit as $t \to \infty$.

Approximate solution by the method of integral relations. Multiplying Eq. (1.2) by $2\pi rc^k(r, t)$ and integrating from 0 to ∞, we obtain

$$\frac{1}{k+1}\ \frac{d}{dt}M_{k+1}+Q\int_0^\infty c^k\frac{\partial c}{\partial r}dr=\int_0^\infty 2\pi Dc^k\frac{\partial}{\partial r}r\frac{\partial c}{\partial r}dr-\alpha M_{k+n}$$

$$M_k=2\pi\int_0^\infty rC^k(r,\ t)dr$$

Hence, we have a system of integral relations

$$\frac{dM_1}{dt}=-\alpha M_n$$

$$\frac{1}{k+1}\ \frac{dM_{k+1}}{dt}=-2\pi Dk\int_0^\infty c^{k-1}r\left(\frac{\partial c}{\partial r}\right)^2dr-\alpha M_{k+n},\ \ k>0$$

We will use the simplest approximation of the required solution, which agrees both with the linear solution and with the data of the numerical experiment

$$c(r,\ t)=A(t)\left(\frac{r}{l}\right)^q\exp\left(-\frac{r^2}{l^2(t)}\right),\ \ q=\frac{Q}{2\pi D}$$

In the case of this approximation, the moments M_k are explicitly calculated and expressed in terms of Γ-function [14]. As a result, the system of equations for finding $A(t)$ and $l(t)$, which uses the first two integral relations, is reduced in the variables $A(t)$, $X(t) = Al^2$ to the form

$$dX/dt=-\alpha\mu_1XA^{n-1},\ \ X=Al^2$$

$$\frac{dA}{dt}=-\frac{4DA^2}{X}+\alpha(\mu_1-\mu_2)A^n \tag{5.4}$$

$$\mu_1=\frac{\Gamma(nq/2)}{n^{nq/2}}\ \frac{1}{\Gamma(q/2)},\ \ \mu_2=\frac{2^{q+1}\Gamma((n+1)q/2)}{(n+1)^{(n+1)q/2}\Gamma(q)}$$

Here, again, the magnitude of X is proportional to the total supply of material in the slug and the magnitude of A is proportional to the "amplitude" of the concentration pulse.

In the (X, A) plane, we have the equation

$$\frac{dA}{dX}=\frac{4D}{\alpha\mu_1}\ \frac{A^{3-n}}{X^2}+\frac{\mu_2-\mu_1}{\mu_1}\ \frac{A}{X} \tag{5.5}$$

As in the case of rectilinear flow, cases when $n < 2$, when all the trajectories pass through the origin of coordinates and cases when $n > 2$. when all the solutions tend to points on the X axis, that is, the slug spreads asymptotically, conserving a finite supply of the substance, are qualitatively different. In the boundary case when $n = 2$, Eq. (5.5) turns out to be linear and the problem is solved in quadratures

$$A=A_0\left(\frac{X}{X_0}\right)^{\mu-1}\exp\left(-\frac{4D}{\alpha\mu_1}\left(\frac{1}{X}-\frac{1}{X_0}\right)\right)$$

$$t=\frac{1}{\alpha\mu_1}\int_X^{X_0}\frac{dX}{XA(X)},\ \ \mu=\frac{\mu_2}{\mu_1} \tag{5.6}$$

Again, the second relation of (5.6) is indicative of the extremely slow decay of the total supply with time.

The approximate solution, obtained by the method of integral relations (the dashed curves) and the results of the numerical calculation (the solid curves) are shown in Fig. 4 in the critical case ($n = 2$) for $q = 6$, $\alpha = 100$ and $t = 1$, 10 and 100, respectively, for curves 1, 2 and 3 where the values of $c(r, t)$ are reduced by a factor of 10 in the case of curves 1 and increased by a factor of 10 in the case of curves 3. A comparison shows good agreement between the approximate, analytical and numerical results.

6. DISPLACEMENT OF A THIN SLUG OF ACTIVE IMPURITY

In estimating the influence of the evolution of the impurity distribution on the process of the petroleum displacement by a solution of an "active impurity", that is, a solution of a substance which is used to increase the petroleum yield (see [1–3, for example]), three simplifying assumptions are made below: (1) the inverse effect of the two-phase nature of the flow is not taken into account, so that the concentration distribution of the impurity is the same as in a single-phase flow (this assumption is acceptable if the solubility of the impurity in the aqueous and petroleum phases is the same), (2) convective transport of the impurity predominates over diffusive transport and therefore the slug size is small compared with the specific size of the stratum (a "narrow slug"), and (3) capillary transport can be neglected.

With these assumptions, the investigation of the displacement reduces to solving the problem [1–3]

$$m\frac{\partial s}{\partial t}+v\,\frac{\partial F(s,\ c)}{\partial x}=0$$

$$s(x,\ 0)=s_0,\quad s(0,\ t)=s^0,\quad c=c(x,\ t) \tag{6.1}$$

Here s is the water saturation, U is the displacement rate, m is the porosity of the stratum and F is a function of the flux distribution (the Buckley–Leverett function) which depends on the concentration of the active impurity as on the parameter. The distribution $c(x, t)$ is assumed to be known (from the solution of the problem considered above, say) and to be concentrated close to the known trajectory of the slug in $x = X_0(t)$. Everywhere outisde of the narrow boundary layer close to the trajectory of the slug, the saturation distribution is governed by the Buckley–Leverett equation (Eq. (6.1) with $F_0(s) = F(s, 0)$).

The thinness of the slug enables us to use the procedure of matched asymptotic expansions as was done previously with reference to similar problems [3, 5]. On introducing the inner coordinate $\xi = x - X_0$ in the boundary layer, we find that the saturation distribution in the transition zone satisfies the relation

$$UF(s,\ c)-mVs=C,\quad V=X_0'(t) \tag{6.2}$$

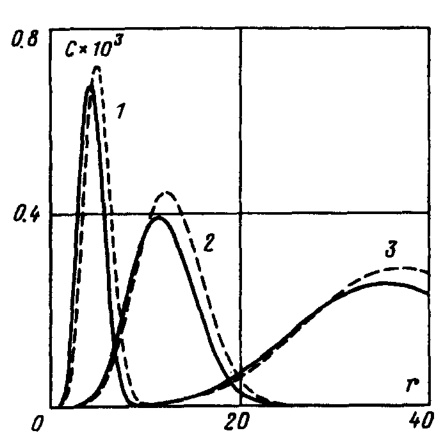

Fig. 4.

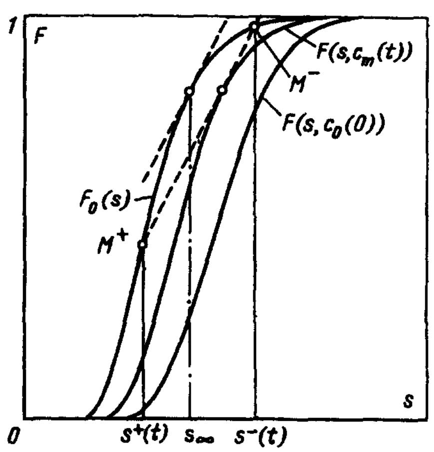

Fig. 5.

at each instant of time, where C and V are slowly varying functions of t. Relation (6.2) defines a straight line in the (s, F)-diagram (Fig. 5), the points of intersection of which with the graph $F_0(s)$, M^+ and M^- correspond to states behind the slug and in front of it at a given instant. The slope of this line is known and it is equal to mV/U. However, the free term $C(t)$ remains undetermined. In order to determine it, we note that, at a fixed t within the transition layer, we have a humped-shaped concentration profile $C(x, t)$. It follows from (6.2) that

$$-mV\frac{\partial s}{\partial \xi}+U\frac{\partial F}{\partial s}\ \frac{\partial s}{\partial \xi}+U\frac{\partial F}{\partial c}\ \frac{\partial c}{\partial \xi}=0$$

However, when $\xi = 0$, we have $\partial c/\partial \xi = 0$, $c = c_{\max}(t)$ such that $V = (U/m)\partial F(s, c_{\max})/\partial s$. This means that, at each instant t, the line (6.2) in the (s, F)-diagram touches the graph of $F(s, c)$ as a function of s with $c = c_{\max}(t)$ (Fig. 5). The actual values $s^+(t)$ and $s^-(t)$ of the saturation on both sides of the trajectory of the slug $X_0(t)$ are uniquely defined by this.

After this the saturation distribution $s(x, t)$ is found by the method of characteristics and is given by the expressions

$$x(s,\ t)=F_0'(s)\frac{U}{m}t,\quad s\leqslant s_0^+,\quad s\geqslant s_0^-$$

$$x(s,\ t)=F_0'(s)\frac{U}{m}(t-t')s^{\mp}(t')+X_0(t')$$

(the minus superscript is taken when $s_\infty \leqslant s^- \leqslant s_0^-$ and the plus superscript when $s_0^+ \leqslant s \leqslant s_\infty$). Here s_∞ is the limiting value of the saturation in front of the decaying slug, which is given by the solution of the equation $V = (U/m)F_0'(s_\infty)$. Hence, in the case under consideration, all the technical characteristics which are of interest can be calculated explicitly.

In concluding, we note that the same approach can be applied without any changes to axially symmetric radial flow, if one puts $x = r^2/2$, where r is the distance from the pore.

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